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AN IN-DEPTH ANALYSIS OF OSCILLATIONS IN FRACTIONAL VECTOR PDES: QUANTITATIVE AND QUALITATIVE PERSPECTIVES

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Abstract: *Fractional differential equations have gained significant attention for modeling complex processes across various fields such as porous structures, electrical networks, and industrial robotics. They offer a versatile framework for understanding phenomena with self-similar properties, viscoelasticity, and more. This paper delves into the study of oscillatory solutions, a crucial aspect of fractional differential equations, shedding light on their quantitative and qualitative characteristics.*

While oscillatory behavior in scalar fractional ordinary differential equations has received some attention in previous research, this paper extends the analysis to scalar fractional partial differential equations, a less-explored area. By exploring oscillations in this broader context, we contribute to a deeper understanding of complex processes modeled by fractional differential equations.

Keywords: *Fractional differential equations, oscillatory behavior, partial differential equations, qualitative analysis, quantitative analysis.*

1 Introduction

Fractional differential equations are now recognized as an excellent source of knowledge in modelling dynamical processes in self similar and porous structures, electrical networks, probability and statistics, visco elasticity, electro chemistry of corrosion, electro dynamics of complex medium, polymer rheology, industrial robotics, economics, biotechnology etc. See the recent monograph [2, 11-14, 16, 23, 29] for theory and applications of fractional differential equations. Oscillatory solution plays an important role in the quantitative and qualitative theory of fractional differential equations. There are several papers dealing with oscillation of scalar fractional ordinary differential equations [3-5, 9, 24, 27-28]. However, only a few results have appeared regarding the oscillatory behavior of scalar fractional partial differential equations, see [1, 18-22, 26] and the references cited there in.

In 1970, Domslak introduced the concept of H-oscillation to investigate the oscillation of solutions of vector differential equations, where H is a unit vector in R^n . We refer the articles [6-7] for vector ordinary differential equations and [8, 15, 17, 25] for vector partial differential equations. To the present time, there exists almost no literature on oscillation results for vector fractional ordinary differential equations and vector fractional partial differential equations, particularly for vector fractional nonlinear partial differential equations. Motivated by this, we initiate the fractional order vector partial differential equations for delay equations.

Formulation of the problems: The oscillatory theory of fractional differential equation was introduced by

Grace et al [9]

$D_a^q x \square f_1(t,x) = v(t) \square f_2(t,x)$ $\lim J^1{}_a{}^{\square q} x(t) = b,$

$t \square a^{\square}$

where D_a^q denotes the Riemann-Liouville differential operator of q , where $0 < q < 1$.

Chen [4] and Han et al [28] studied the oscillation of the fractional differential equation with Liouville right sided fractional derivative of order □ of the following form

□ □□ □□ □$q(t) f$ □□□□$(s$ □$t)$□□$y(s)ds$□□ = 0, $t > 0$, □□$r(t)$ D□□ y (t)□

□ □ □ t □

□ □□$y(t)$□□□ □ $p(t) f$ □□□□ $(s$ □$t)$□□$y(s)ds$□□ = 0, $t > 0$. $r(t)g(D$□

□ t □

Prakash et al. [18] and Sadhasivam and Kavitha [21] investigated the fractional partial differential equation with Riemann-Liouville left sided definition on the half axis $R_{\square}$ of the form

□ □ □ □ □□t □□ □

$r(t)D_{\square,t}u(x,t)$ □ $q(x,t) f$ □ $(t$ □ $v)$ $u(x,v)dv$□ = $a(t)$□$u(x,t)$, (x,t) □□□ $R_{\square} = G$,

□t □ 0 □

with the Neumann boundary condition

□$u(x,t)$

= 0, (x,t)□□□□ $R_{\square}$.

□N

□ □ □m □□t □□ □

□ □

$p(t)g(D_{\square,t}u(x,t))$ □ $q_j(x,t) f_j$ □ $(t$ □ $s)$ $u(x,s)ds$□ = $a(t)$□$u(x,t)$ □ $F(x,t)$, (x,t)□□□$R_{\square} = G$,

□t $j=1$ □ 0 □

subject to the boundary condition

□$u(x,t)$

□□$(x,t)u(x,t) = 0$, (x,t)□□□□ $R_{\square}$.

□□

To the best of our knowledge, nothing is known regarding the H-oscillatory behavior for the following class of vector fractional partial differential equations with forced term of the form

m

$D^{\square}{}_{\square,t}$ □$r(t)D^{\square}{}_{\square,t}U(x,t)$□= $a(t)$□$U(x,t)$ □ □$a_i(t)$□$U(x,$□$_i(t))$

$i=1$

k t □ ‖ ‖

□ □□

□ □$p_j(x,t) f_j$ □□ $(t$ □ $s)U(x,$□$_j(s))$ ds□$U(x,$□$_j(t))$

□ 0□

$j=1$

□$F(x,t)$, (x,t)□G = □□$R_{\square}$,

$R_{\square} = (0,\square)$, where □ is a bounded domain in R^n with a piecewise smooth boundary □□,□□(0,1) is a constant, $D_{\square}{}^{\square}{}_{,t}$ is the Riemann-Liouville fractional derivative of order □ of u with respect to t , □ is the Laplacian

2

n n □ $u(x,t)$ ‖

operator in the Euclidean n - space R (ie) □$u(x,t)$ = □ 2 and $U(x,$□$_j$ ‖ (s)) is the usual Euclidean norm in
r=1 □$_x r$
R^n .

Equation (1.1) is supplemented with the following boundary conditions
□$U(x,t)$
□□$(x,t)U(x,t) = 0$, (x,t)□□□□R□, (1.2)

□□
where □ is the unit exterior normal vector to □□ and □(x,t) is positive continuous function on □□□ R□ and
$U(x,t) = 0$, (x,t)□□□□R□. (1.3)
In what follows, we always assume without mentioning that
(A_1) $r(t)$□C□(R□;R□),$a(t),a_i(t)$□C(R□;R□),i =1,2,...m ;
(A_2) □$_j$,□$_i$□C(R□;R), lim□$_j(t)$ = lim□$_i(t)$ = □,i =1,2,...m, j =1,2,...,k ;
t□□ t□□
(A_3) p_j□$C(G;R)$ and $p_j(t) = min_x$□□ $p_j(x,t)$, j□I_k = □1,2,...,k□;
(A_4) F□$C(G;R^n)$, $f_H(x,t)$□$C(G;R)$ and □ $f_H(x,t)dx$ □ 0;
□
(A_5) f_j□C(R□;R) are convex and non decreasing in R with $uf_j(u) > 0$ for u □ 0 and there exist positive
$f_j(u)$

constants □$_j$ such that □□$_j$ for all u □ 0, j □I_k. u
The study of H-oscillatory behavior of fractional partial differential equation is initiated in this paper. Our approach is to reduce multi-dimensional problems for (1.1) to one dimensional oscillation problems for scalar functional fractional differential inequalities. The purpose of this paper is to establish some new H-oscillation criteria for equation (1.1) with (1.2) and equation (1.1) with (1.3) by using a generalized Riccati technique and integral averaging method. Our results are essentially new.

2 Preliminaries

In this section, we give the definitions of H-oscillation, fractional derivatives and integrals and some notations which are useful throughout this paper. There are serveral kinds of definitions of fractional derivatives and integrals. In this paper, we use the Riemann-Liouville left sided definition on the half-axis R□. The following notations will be used for the convenience.
$u_H(x,t) = \langle U(x,t),H\rangle$, $f_H(x,t) = \langle F(x,t),H\rangle$,
1
$V_H(t)$ = □$|u_H(x,t)dx$, where □ = □ dx. (2.1)
□□ □

Definition: 2.1 By a solution of (1.1), (1.2) and (1.3) we mean a non trivial function
$U(x,t)$□C^{2}□$(G;R^n)$□$C^2(G$ □[$\hat{t}$□$_1$,□);R^n)□$C(G$ □[$\tilde{t}$□$_1$,□);R^n) and satisfies (1.1) on G and the boundary conditions
(1.2) and (1.3), where $\hat{t}$□$_1$ = min□□0, min □□inf □$_i(t)$□□□□ , $\tilde{t}$□$_1$ = min□□0, min □□inf □$_j(t)$□□□
□ .
□ 1□i□m □t□0 □□ □ 1□ j□m □t□0 □□

Definition: 2.2 Let H be a fixed unit vector in R^n. A solution $U(x,t)$ of (1.1) is said to be H-oscillatory in
G if the inner product $\langle U(x,t),H\rangle$ has a zero in □□$(t,$□) for any $t > 0$. Otherwise it is H-nonoscillatory.

Definition: 2.3 The Riemann-Liouville fractional partial derivative of order 0 <□<1 with respect to t of a
function $u(x,t)$ is given by

$D^{\square}_{\square,t}u(x,t) := \square \frac{1}{\square t \square(1\square\square)} \square^{t}_{0} (t \square v)^{\square\square} u(x,v)dv,$ (2.2)

provided the right hand side is pointwise defined on $R_{\square}$ where $\square$ is the gamma function.

Definition: 2.4 The Riemann-Liouville fractional integral of order $\square > 0$ of a function $y : R_{\square} \square R$ on the half-axis $R_{\square}$ is given by

$I_{\square} y(t) := \frac{1}{\square(\square)} \square^{t}_{0} (t \square v)^{\square\square 1} y(v)dv \quad for \quad t > 0,$ (2.3)

provided the right hand side is pointwise defined on $R_{\square}$.

Definition: 2.5 The Riemann-Liouville fractional derivative of order $\square > 0$ of a function $y : R_{\square} \square R$ on the half-axis $R_{\square}$ is given by

$D^{\square}_{\square} y(t) := \frac{d^{\square\square\square}}{dt^{\square\square\square}} \square I_{\square}^{\square\square\square\square\square} y \square(t) \quad for \quad t > 0,$ (2.4)

provided the right hand side is pointwise defined on $R_{\square}$ where $\square\square\square$ is the ceiling function of $\square$. where m is a positive integer.

Lemma: 2.1 [11] Let y be solution of (1.1) and

$K(t) := \square^{t}_{0} (t \square s)^{\square\square} y(s)ds \quad for \ \square\square(0,1) \ and \quad t > 0.$ (2.5)

Then

$K\square(t) = \square(1\square\square) D^{\square}_{\square} y(t) for \ \square\square(0,1) \ and \quad t > 0.$ (2.6)

Lemma: 2.2 [10] If X and Y are nonnegative, then

$mXY^{m\square 1} \square X^{m} \square (m\square 1)Y^{m},$ (2.7)

3 **H-Oscillation of the problem** (1.1),(1.2)

We begin with the following Lemma.

Lemma: 3.1 Assume that $(A_1) \square (A_5)$ hold. Let H be a fixed unit vector in R^n and $U(x,t)$ be a solution of (1.1) . (i)If $u_H(x,t)$ is eventually positive, then $u_H(x,t)$ satisfies the scalar fractional partial inequality

$D^{\square}_{\square,t} \square r(t) D^{\square}_{\square,t} u_H(x,t) \square\square \ a(t) \square u_H(x,t) \square \sum_{i=1}^{m} \square a_i(t) \square u_H(x,\square_i(t))$

$\square \sum_{j=1}^{k} \square\square p_j(t) f_j \square\square \square^{t}_{0} (t \square s)^{\square\square} u_H(x,\square_j(s))ds \square u_H(x,\square_j(t)) \square \square f_H(x,t).$ (3.1)

(ii)If $u_H(x,t)$ is eventually negative, then $u_H(x,t)$ satisfies the scalar fractional partial inequality

$D^{\square}_{\square,t} \square r(t) D^{\square}_{\square,t} u_H(x,t) \square\square \ a(t) \square u_H(x,t) \square \sum_{i=1}^{m} \square a_i(t) \square u_H(x,\square_i(t))$

$\square \sum_{j=1}^{k} \square\square p_j(t) f_j \square\square \square^{t}_{0} (t \square s)^{\square\square} u_H(x,\square_j(s))ds \square u_H(x,\square_j(t)) \square \square f_H(x,t).$ (3.2)

Proof. Let $u_H(x,t)$ be eventually positive. Taking the inner product of (1.1) and H, we get

m

$D^{□}_{□,t}$ □$r(t)D^{□}_{□,t}\langle U(x,t),H\rangle$ □= $a(t)$□$\langle U(x,t),H\rangle$ □ $\langle$□$a_i(t)$□ $\rangle$ $U(x,□_i(t)),H$

i=1

k t □

□

□□$p_j(x,t)f_j$□□ (t □ s) $U(x,□_j(s))\,ds$□ $U(x,□_j(t)),H$ □ $F(x,t),H$,

j=1 □ 0 □

that is,

m

$D^{□}_{□,t}$ □$r(t)D^{□}_{□,t}u_H(x,t)$□= $a(t)$□$u_H(x,t)$ □ □$a_i(t)$□$u_H(x,□_i(t))$

i=1

k

□

□□$p_j(x,t)f_j$□□$t(t$ □ $s)$□□$U(x,□_j(s))\,ds$□ □$u_H(x,□_j(t))$□ $f_H(x,t)$. (3.3)

j=1 □ 0□

By (A_3), we have

□ t □□□ $p_j(x,t)f_j$ □□ (t□ s)$U(x,□_j(s))$ ds□$u_H(x,□_j(t))$

□ 0□

□ t □□□

□ $p_j(t)f_j$□□ (t □ s)$U(x,□_j(s))$ ds□$u_H(x,□_j(t))$,

□ 0□

since f_j□$C(R_□,R)$, j =1,2...k, we have $u_H(x,□_j(s))$ □ $\|U(x,□_j(s))\|$, therefore

□ t □□□ $p_j(t)f_j$ □□ (t□ s)$U(x,□_j(s))$ ds□$u_H(x,□_j(t))$

□ 0□

□ t □□ □

□ $p_j(t)f_j$□□ (t □ s) $u_H(x,□_j(s))ds$□$u_H(x,□_j(t))$, j =1,2,...,k. (3.4)

□ 0 □

Using (3.4) in (3.3), we get

m

$D^{□}_{□,t}$ □$r(t)D^{□}_{□,t}u_H(x,t)$□□ $a(t)$□$u_H(x,t)$ □ □$a_i(t)$□$u_H(x,□_i(t))$

i=1

k t □

□ □□

□□$p_j(t)f_j$□□ (t □ s) $u_H(x,□_j(s))ds$□$u_H(x,□_j(t))$ □ $f_H(x,t)$.

□ 0 □ j=1

Similarly, let $u_H(x,t)$ be eventually negative, we easily obtain (3.2). The proof is complete.
The inner products of (1.2),(1.3) with H yield the following boundary conditions.

□$u^{H(x,t)}$ □□$(x,t)u_H(x,t)$ = 0, (x,t)□□□□$R_□$, (1.2)□

□□

$u_H(x,t)$ = 0, (x,t)□□□□$R_□$. (1.3)□

Lemma: 3.2 Assume that (A_1) □(A_5) hold. Let H be a fixed unit vector in R^n. If the scalar fractional partial inequality (3.1) has no eventually positive solutions and the scalar fractional partial inequality (3.2) has no eventually negative solutions satisfying the boundary conditions (1.2)□ or (1.3)□, then every solution $U(x,t)$ of the problem (1.1),(1.2) or (1.1),(1.3) is H-oscillatory in G. Proof. Suppose to the contrary that there is a H-nonoscillatory solution $U(x,t)$ of (1.1),(1.2) or (1.1),(1.3) in G, then $u_H(x,t)$ is eventually positive or $u_H(x,t)$ is eventually negative. If $u_H(x,t)$ is eventually positive, then by Lemma 3.1 $u_H(x,t)$ satisfies the boundry condition (1.2)□ or (1.3)□ . This contradicts the hypothesis. The similar proof follows when $u_H(x,t)$is eventually negative. **Theorem: 3.1** Assume that (A_1) □(A_5) and (A_6) $\min_{j□I}$□□$_j(t)$□=□(t) □ t .

K
(A_7) $u_H(x,t)$ □ L hold . If the fractional differential inequality

D□□ □$r(t)D$□□$V_H(t)$□□ L□$p_j(t) f_j(K_H(t))$ □ 0, (3.5) k
$j=1$ $j=1$
has no eventually positive solutions and the fractional differential inequality has no eventually negative solutions, then every
k

D□□□$r(t)D$□□$V_H(t)$□□ L□$p_j(t) f_j(K_H(t))$ □ 0, (3.6)

solution $U(x,t)$ of (1.1) and (1.2) is H-oscillatory in G .
Proof. Suppose to the contrary that there exists a solution $U(x,t)$ of (1.1) , (1.2) which is not a H-oscillatory in G . Without loss of genearality, we may assume that $u_H(x,t) > 0$ in □□[t_0,□) for some $t_0 > 0$. Integrating (3.1) with respect to x over □ , we obtain

m
□ D□□ □$r(t)D$□□$u_H(t)$□dx □ $a(t)$□ □$u_H(x,t)dx$ □ □$a_i(t)$□ □$u_H(x,$□$_i(t))dx$
□ □ □
$i=1$
□
k t □□ □
□□$p_j(t)$□f_j□□ $(t$ □ $s)$ $u_H(x,$□$_j(s))ds$□$u_H(x,$□$_j(t))dx$ □ □$f_H(x,t)dx$, t □ t_0. (3.7)
$j=1$ □ □ 0 □ □

Using Green's formula and boundary condition (1.2)□ yield that
□u (x,t)
□ □$u_H(x,t)dx$ = □ H dS = □□ □$(x,t)u_H(x,t)dS$ □ 0, t □ t_0 (3.8)
□ □□ □□ □□
and
□ □$u_H(x,$□$_i(t))dx$ = □ □u $H(x,$□$_i(t))$ dS = □□ □$(x,t)u_H(x,$□$_i(t))dS$ □ 0,
□ □□ □□ □□
$i=1,2,\ldots m$, t □ t_0. (3.9)

By using Jensen's inequality (A_6),(A_7) and (2.1) , we get
□ t □□ □
□ f_j□□ $(t$ □ $s)$ $u_H(x,$□$_j(s))ds$□$u_H(x,$□$_j(t))dx$
□ □ 0 □
□ Lf_j□□□□ □□□□□□$_0t$ $(t$ □ $s)$□□$u_H(x,$□$_j(s))ds$□□□dx□□ □□
□ Lf_j□□□$t(t$ □ $s)$□□□□ □ uH $(x,$□j $(s))dx$□□ds□ □
□ 0 □ □ □ □
□ L□ dxf_j□□□$t(t$ □ $s)$□□□ □□ uH $(x,$□j $(s))dx($□ $dx)$□1 □□ds□ □
□ □ 0 □ □ □ □ □
□ t □□ □
□ L□ dxf_j□□ $(t$ □ $s)$ $V_H($□$_j(s))ds$□
□ □ 0 □
□ L□ $dxf_j(K_H(t))$ t □ t_0. (3.10)
□

Also by (A_4) ,
□ $f_H(x,t)dx$ □ 0. (3.11)
□

In view of (2.1), (3.8)-(3.11), (3.7) yield
k

$D^{\square}_{\square} \square r(t) D^{\square}_{\square} V_H(t) \square\square L \square p_j(t) f_j(K_H(t)) \square 0.$ (3.12)

$j=1$

Therefore, $V_H(t)$ is an eventually positive solution of (3.5). This contradicts the hypothesis. The case where $u_H(x,t) < 0$ in $\square\square[t_0,\square)$ can be treated similarly and we are also getting a contradiction. The proof is now complete. **Theorem: 3.2** Suppose that the conditions $(A_1) \square (A_7)$ and

$\square\square$ 1 $\square$

$\square t_0 \quad \square\square\square \quad r(s) \square\square_{\square} ds = \square$ (3.13)

hold

Futhermore, assume that there exists a positive function $\square\square C^{\square}((0,\square);R_{\square})$ such that

$\square\square\square \sim \square k \,\tilde{}\, p_j(s)^{\square} \sim r(s) \square\square \sim \square(s\sim)\square 2 \quad \square \square ds = \square,$ (3.14) limsup $\square L \square(s) \square_j$

$\square\square\square \; \square_1 \square\square \quad j=1 \quad 4\square(1\square\square)\square(s)^{\square}_{\square}$

where $\square_j$ are defined as in (A_5). Then every solution of $U(x,t)$ of the problem (1.1),(1.2) is H-oscillatory in G.

Proof. Suppose to the contrary that there exists a solution $U(x,t)$ of the problem (1.1),(1.2) which is not Hoscillatory in G. Without loss of generality we may assume that $u_H(x,t) > 0$ in $\square\square[t_0,\square)$ for some $t_0 > 0$.

That is, $V_H(t)$ is an eventually positive solution of (3.5). Then there exists $t_1 \square t_0$ such that $V_H(t) > 0$ and $K_H(t) > 0$ for $t \square t_1$. Therefore, it follows from (3.5) that

k

$D^{\square}_{\square} \square r(t) D^{\square}_{\square} V_H(t) \square\square \square L \square p_j(t) f_j(K_H(t)) < 0 \quad for \quad t \square [t_1,\square).$ (3.15)

$j=1$

Thus $D^{\square}_{\square} V_H(t) \square 0$ or $D^{\square}_{\square} V_H(t) < 0, t \square t_1$ for some $t_1 \square t_0$. We now claim that

$D^{\square}_{\square} V_H(t) \square 0, \quad for \quad t \square t_1.$ (3.16)

Suppose not, then $D^{\square}_{\square} V_H(t) < 0$ and there exists $t_2 \square [t_1,\square)$ such that $D^{\square}_{\square} V_H(t_2) < 0$. Since $r(t) D^{\square}_{\square} V_H(t)$ is strictly decreasing on $[t_1,\square)$. It is clear that

$r(t) D^{\square}_{\square} V_H(t) < r(t_2) D^{\square}_{\square} V(t_2) := \square c,$

where $c > 0$ is a constant for $t \square [t_2,\square)$. Therefore from (2.6), we have

$K_H \square (t) \quad \square \quad \square \square c \square$

$= D_{\square} V_H(t) < \square\square \; r(t) \; \square\square \; \square \quad for \quad t \square [t_2,\square).$

$\square(1\square\square) \quad \square$

Then, we get

$\square \; 1 \; \square K_H \square \; (t)$

$\square\square\square \quad r(t) \; \square\square\square \; \square \; \square \; c\square(1\square\square) \quad for \quad t \square [t_2,\square).$

Integrating the above inequality from t_2 to t, we have

$t \square \; 1 \; \square \quad K(t) \square K \quad (t)$

$\square$

$\square_{t2} \; \square\square_{\square} \; r(s) \; \square\square ds \; \square \; \square \quad {}^{H} c\square(1\square\square^{H}) \quad 2$

$< K H(t2) \quad for \quad t \square [t2,\square).$

$c\square(1\square\square)$ Letting $t \square\square$, we get

$ds \square \; \square\square 2 \quad \square\square^{\square} r(1s) \; \square\square^{\square}_{\square} cK\square(1H(\square t2\square)) \; < \square. \; t\square$

This contradicts (3.13). Hence $D^{\square}_{\square} V_H(t) \square 0$ for $t \square [t_1,\square)$ holds. Define the function $W(t)$ by the generalized Riccati substitution

$r(t) D^{\square\square} V^H(t) \quad for \quad t \square [t_1,\square).$ (3.17)

$W(t) = \square(t)$

$K_H(t)$

Then we have $W(t) > 0$ for t □[t_1,□). From (2.6),(2.7), (3.5)and (A_5) it follows that

D□□$W(t)$ = □(t) D□□□$r(t)D$□□$VH(t)$□□ D□□ □□ □(t) □□$r(t)D$□□$VH(t)$

$K_H(t)$ □$K_H(t)$□

□ □□$(t)L$□k $p_j(t) f_j(KH(t))$ □ □$KH(t)D$□□□(t) □□$(t)D$□□ $KH(t)$ □$r(t)D$□□$VH(t)$

$KH(t)$ □□□ $KH2(t)$ □□□

j=1

k □ □

□ □L□(t)□□$_j$ $p_j(t)$ □ D □□(t)$W(t)$ □ D□ $KH(t)$ $W(t)$. (3.18)

j=1 □(t) $KH(t)$

Let $W(t) = \tilde{W}$(□),□(t) =□~ (□), $p_j(t) = \tilde{p}_j$(□),$K_H(t) = \tilde{K}_H$(□) .

Then D□□$W(t) = \tilde{W}$□(□), D□□□(t) =□~ □(□) . Then the above inequality becomes k ~ ~

~ ~ □ ~pj (□)□ □~□(□)$\tilde{W}$(□)□ $\tilde{K}H$□ (□)$\tilde{W}$ (□)

W□(□) □ □L□(□) □j

□(□) K(□)

j=1 H

□ □L□~(□)□j^k=1□ ~p_j(□)□□□□~~□(□)$\tilde{W}$(□)□ □(1□ ~□□)~$r\tilde{W}$ (□²) (□) . (3.19) j

(□) (□)

Using Lemma 2.2 and (3.20) in (3.19), we have □(1□□) ~

1

k ~r(□)□□~ □(□)□² ~r(□)

~ ~ ~

W□(□) □ □L□(□)□□$_j$ ~p_j(□) □ 1 ~ . (3.21)

Taking m = 2,

X = ~(□) ~r (□)W(□), Y = 2 □(1□□)□~(□) □□(□). (3.20) □

j=1 4 □(1□□)□(□)

Integrating both sides of the above inequality from □$_1$ to □, we obtain □□□□L□~(s)□k □ ~pj (s)□ 1 ~r (s)□□~ □(~s)□2 □□ds □$\tilde{W}$(□1)□$\tilde{W}$(□) < $\tilde{W}$ (□1).

j

1 □ j=1 4 □(1□□)□(s)□ □

□ □

Taking the limit supremum of both sides of the above inequality as □□□, we get

limsup□□□□L□~(s)□k □ ~pj (s)□ 1 ~$r(s)$□□~ □(~s)□2 □□ds < $\tilde{W}$ (□1) < □,

j

□□□ □$_1$ □□ j=1 4 □(1□□)□(s)□□

which contradicts (3.14) and completes the proof.

Theorem: 3.3 Suppose that the conditions (A_1) □(A_7) and (3.13) hold. Futhermore, suppose that there exists a positive function □□C□((0,□);R□) and a function P□$C(D,R)$ where D := □(t,s):t □ s □ t_0□ such that

1. $P(t,t) = 0$ for t □ t_0 ,

2. $P(t,s) > 0$ for (t,s)□D_0, where D_0 := □(t,s):$t > s$ □ t_0□ and P has a continuous and non-positive □$P(t,s)$
partial derivative P_s□(t,s) = on D_0 with respect to the second variable and satisfies □s

limsup 1 □□P(□,s)□□L□~(s)□k □ ~p_j (s)□ 1 r~(s)□□~ □(~s)□2 □□ds = □, (3.22)

j
□□□ P(□,□₁)□ □□ j=1 4 □(1□□)□(s)□□

1
where □$_j$ are defined as in Theorem 3.2. Then all the solutions of $U(x,t)$ of the problem (1.1),(1.2) is H-oscillatory in G . Proof. Suppose that $U(x,t)$ is H-nonoscillatory solution of (1.1),(1.2) . Without loss of generality we may assume that $u_H(x,t)$ is an eventually positive solution . Then $V_H(t)$ is an eventually positive solution of (3.5). Then proceeding as in the proof of Theorem 3.2, to get (3.21)

W~□(□) □ □L□~(□)□k □$_j$ ~p_j (□) □ 1 r~(□)□□~ □(□~)□2 ,

j=1 4 □(1□□)□(□)

multiplying the previous inequality by P(□,s) and integrating from □₁ to □ for □□[□₁,□) , we obtain
□□P(□,s)□□L□~(s)□k □ ~p_j (s)□ 1 r~ (s)□□~□(~s)□2 □□ds □ □□P(□,s)W~(s)□□□ □ □□P_s□(□,s)W~ (s)ds j

□ □ j=1 4 □(1□□)□(s)□□ 1 □1
1 □
~ □ ~ ~
□ P(□,□₁)W (□₁)□ □ P_s□(□,s)W(s)ds < P(□,□₁)W(□₁).
□
1
□ Therefore 1 □P(□,s)□□L□~(s)□k □$_j$ ~p_j (s)□ 1 ~r (s)□□~ □(~s)□2 □□ds < W~ (□1) < □,

P(□,□₁)□ □□ j=1 4 □(1□□)□(s)□□

1
which is a contradiction to (3.22).The proof is complete.

Corollary 3.1 Assume that the conditions of Theorem 3.3 hold with (3.22) replaced by
□ k
1 □ ~ □ ~p_j (s)ds = □, limsup P(□,s)L□(s) □$_j$

□□□ P(□,□1)□ j =1
1
□
1 ~r(s)□□~ □(s)□2
limsup □P(□,s) ~ ds < □,

□□□ P(□,□₁)□ □(1□□)□(s)
1
then every solution $U(x,t)$ of (1.1),(1.2) is H-oscillatory in G . Next, we consider the case
□ 1

□ ds < □, (3.23)

t_0 $r(s)$

which yields that (3.13) does not hold. In this case, we have the following result.

Theorem: 3.4 Suppose that the conditions (A_1) □(A_7) and (3.23) hold and that there exists a positive function □□C□((0,□);R□) such that (3.14) holds. Futhermore, assume that for every constant □$_T$ □□$_0$, where

□$_T$ = max□□$_3$,□$_4$□

□ □ $_k$ □ □

□ □□ ~r (1u) □□j □ ~$p\, j$ (s)ds□□du = □. (3.24)

□T □□ j=1 □$_T$ □□

□

~ □□ ~

Then every solution of V_H(□) of (3.5) is H-oscillatory or satisfies lim □□□□ s□ V_H(s)ds = 0. Proof. Suppose

□□□

0 that $U(x,t)$ is H-nonoscillatory solution of (1.1),(1.2) . Without loss of generality we may assume that $u_H(x,t)$ is an eventually positive solution . Then $V_H(t)$ is an eventually positive solution of (3.5). Then proceeding as in the proof Theorem 3.2, there are two cases for the sign of D□□$V_H(t)$. The proof when D□□$V_H(t)$ is eventually positive is similar to that of Theorem 3.2 and hence is omitted. Next, assume that D□□$V_H(t)$ is eventually negative. Then there exists

t_3 □ t_2 such that D□□$V_H(t)$ < 0 for t □ t_3.From (2.6), we get

K□$_H(t)$ = □(1□□)D□□$V_H(t)$ < 0, for t □ t_3.

~ ~ ~ ~

Then K_H□ (□) = □(1□□)V_H□ (□) < 0 for □□□$_3$. Thus we get lim K_H(□):= M_1 □ 0 and K_H(□) □ M_1. We claim that

□□□

M_1 = 0 . Assume not, that is, M_1 > 0 then from (A_5) , we get

k

D□□ □$r(t)D$□□$V_H(t)$□□ □L□$p_j(t)\, f_j$□$K_H(t)$□

j=1

k

□ □LM_1□□$_j\, p_j(t)$, for t □[t_3,□).

j=1

Let $r(t)$ = ~r (□),$V_H(t)$ = V~$_H$(□), $p_j(t)$ = ~p_j(□) .

Then D□□$V_H(t)$ = V~$_H$□ (□),D□□ □$r(t)D$□□$V_H(t)$□= □~r(□)V~$_H$□ (□)□□ .

Using these values, the above inequality becomes

□~r(□)V~$_H$□ (□)□□ □ □LM_1□^{k} □$_j$ ~p_j (□), for □□[□$_3$,□). Integrating both sides of the last inequality from □$_3$ to □, we have

j=1

□□□ ~ □□ 1 □k □□~pj (s)ds

~r (s)VH□ (s) ds □ □LM □j

□3 j=1 □3

k □ k □

~r (□)V~$_H$□ (□) □ r(□$_3$)V~$_H$□ (□$_3$) □ LM_1□□$_j$□ ~p_j (s)ds □ □k_1 □ LM_1□□$_j$□ ~p_j (s)ds

j=1 □ 3 j=1 □3

k □

k □ K~□ □ LM1□□j □ ~p j (s) ds
~p (s)ds.
□ □LM1□□ j □ j Hence from (2.6), we get H (□) =V~H□ (□) □ j=1~r(□) □ .

j=1 □3 □(1□□)
k □
□□j □ ~p j (s)ds
~ □ j=1 □3
Integrating the last inequality from □4 to □, we get K_H(□) □ K_H(□4)□□(1□□)LM_1□ 4 ~r (u) du.

□
~ ~
Letting □□□, from (3.24), we get lim K_H(□) = □□. This contradicts K_H(□) > 0. Therefore we have M_1 = 0, that
□□□
~ □ □□ ~ is, lim K_H(□) = 0. That is, lim □ (□□s) V_H(s)ds = 0 . Hence the proof.
□□□ □□□ 0

4 H-Oscillation of the problem (1.1),(1.3)

In this section we establish sufficient conditions for the oscillation of all solutions of (1.1),(1.3). For this we need the following:The smallest eigen value □0 of the Dirichlet problem. □□(x)□□□(x) = 0 in □, □(x) = 0 on □□, is positive and the corresponding eigen function □(x) is positive in □.

Theorem: 4.1 Let all the conditions of Theorem 3.2 and 3.3 be hold. Then every solution of U(x,t) of (1.1) and (1.3) H-oscillates in G . Proof. Suppose that U(x,t) is a H-nonoscillatory solution of (1.1) and (1.3) . Without loss of generality we may assume that u_H (x,t) > 0, in □□[t_0,□) for some t_0 > 0. Multiplying both sides of the Equation (3.1) by □(x) > 0 and then integrating with respect to x over□.,
m
we obtain for t □ t_1, □ D□□ □r(t)D□□u_H(x,t)□□(x)dx □ a(t)□ □u_H(x,t)□(x)dx □ □a_i(t)□ □u_H(x,□i(t))□(x)dx
□ □ □
i=1
□
k t □□ □
□□p_j(t)□f_j□□ (t □ s) u_H(x,□j(s))ds□u_H(x,□j(t)□(x)dx □ □ f_H(x,t)□(x)dx. (4.1)
□ □ 0 □ □
j=1
Using Green's formula and boundary condition (1.3)□ it follows that
□ □u_H(x,t)□(x)dx = □ u_H(x,t)□□(x)dx = □□0 □ u_H(x,t)□(x)dx □ 0, t □ t_1 (4.2)
□ □ □
and
□ □u_H(x,□i(t))□(x)dx = □ u_H(x,□i(t))□□(x)dx = □□0 □ u_H(x,□i(t))□(x)dx □ 0,
□ □ □
t □ t_1, i =1,2,...m. (4.3)
By using and Jensen's inequality, (A_6) and (A_7) we get □ f_j □□□t (t □ s)□□u_H(x,□j(s))ds□□ u_H(x,□j(t))□(x)dx

□ □ 0 □
$(t □ s)$ □ □ $Lf j$ □□□□□□□□□□0t(t □ s)□□uH (x,□j (s))□(x)ds□□□□dx□ □□□
□ $Lf j$ □□□□0t □□□□□□□uH (x,□j (s))□(x)dx□□□□ds□ □□
□
□ □□t □□□□ $u_H(x,□_j(s))$□(x)dx(□□(x)dx)□1 □□ds□□. Set
□ L □(x)dxf_j□ (t □ s) □
□ □ 0 □ □ □ □ □
□1

$V_H(t)$ = □ $u_H(x,t)$□(x)dx□□□□(x)dx□□ , t □ t_1. (4.4)

□ □ □ □ □ t □□ □

Therefore, □ f_j □□ (t □ s) $u_H(x,□_j(s))$ds□$u_H(x,□_j(t))$□(x)dx □ L□□(x)dx$f_j(K_H(t))$, t □ t_1, j □I_m. (4.5)
□ □ 0 □ □
By (A_5) , □ $f_H(x,t)$□(x)dx □ 0. (4.6)
□
k
In view of (4.4), (4.2)-(4.6), (4.1) yields D□□ □r(t)D□□$V_H(t)$□□ L□$p_j(t) f_j(K_H(t))$ □ 0, (4.7)
j=1 for t □ t_1. Rest of the proof is similar to that of Theorems 3.2 and 3.3, and hence the details are omitted.

Corollary 4.1 If the inequality (4.7) has no eventually positive solutions, then every solution $U(x,t)$ of (1.1) and (1.3) is H-oscillatory in G .

Corollary 4.2 Let the conditions of Corollary 3.1 hold; then every solution $U(x,t)$ of (1.1) and (1.3) is Hoscillatory in G .
~
Theorem: 4.2 Let the conditions of Theorem 3.4 hold; Then every solution V_H(□) of (4.7) is H-oscillatory
□
□□ ~ or satisfies lim □□□□ s□ $V_H(s)ds$ = 0. The proofs of Corollaries 4.1 and 4.2 and Theorems 4.2 are similar to that of in
□□□
0
Section 3 and hence the details are omitted.

5 Examples

In this section we give an example to illustrate the results established in Sections 3. **Example 1.** Consider the vector fractional partial differential equation
□ □

D□13,t □□t 23 D□13,tU(x,t)□□□ = 1t 23□U(x,t) □ □□ 2□ t31 □ 3 t32 □ □□U(x,t □□)
□□ □□ 4 □□ □(1)2 4 □ □
□ 3 □1 □
□ 1 □□ □ □□t□t □ s□ 3 U□□x,s □ □□□ ds□□□ U□□□x,t □ □2 □□□ □ F(x,t), (5.1)

$\sqrt{3}$ 0□ 2 □ □
(x,t)□G , where G = (0,□)□(0,□)□(0,□) , with the boundary condition
□□□ $u_1(0,t)$ □□ □ = U(□,t) = □□□uu12((□□,,tt))□□□□ = □□□□00 □ □□□, t □ 0.
(5.2)
$U(0,t)$ =
□u_2, (0,t)□ □

Here □ = 1 ,m =1,k =1,n = 2,$r(t) = t$, $p_1(x,t) = 1$, $a(t) = t$, $a_1(t) =$ 2□ $_2$ t 3 □ $3_4 t$ 3 ,□1(t) =□,□1(t) = □ $_2$,

$$F(x,t) = \frac{2\pi}{\sqrt{3}(\Gamma(\frac{1}{3}))^2} t^{\frac{2}{3}} \sin x \cos t$$

and $f_1(u) = u$. It is easy to see that $p_1(t) = \min_{x \in [0,\pi]} p_1(x,t) = \min \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Let $H = e_1 =$ □□₀□□□ , we observe that f_{e1} $(x,t) = \frac{2\pi}{\sqrt{3}(\Gamma(\frac{1}{3}))^2} t^{\frac{1}{3}} \sin x \cos t$ and

$$\int (x,t)dx = \frac{4\pi^{\frac{1}{3}}}{\sqrt{3}(\Gamma(\frac{1}{3}))^2} f_e t \cos t$$

□ 1 □ 0, □ t □.

Take □₁ =1,□₁ =1,□(s) = s. It is clear that conditions (A_1) □(A_7) and (3.13) hold. Therefore,

□□□□L ~(~$p_1(s)$□ ~r $\frac{1}{\sqrt{3}}$ □ (s)□□~□(s~)□2 □□ds = □□□□L s 1 □ □ds □□ as □□□. □ s)□1

□ 4□(1□□)□(s)□ □□
□ □ □ 1□ 4□( )s □
1
□ 3 □

Thus all the conditions of Theorem 3.2 are satisfied. Hence, it follows that every solution $U(x,t)$ of (5.1),(5.2) is e_1 - oscillatory in G. Infact $U(x,t)$ = □sin $x$sin t□ □□ 3 □□□ , is one such solution of the problem (5.1) and (5.2). We note that the above solution $U(x,t)$ is not e_2 □ oscillatory in G , where e_2 = □□□ ₁□□□.

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