Journal of Statistical and Mathematical Sciences
ISSN 3065-100X
Volume 11 Issue 1
January – March 2023
Impact Factor: 6.80
Enquiry: contact@continentalpub.online
Published by Continental Publication | http://continentalpub.online/index.php/jsms

A COMPREHENSIVE ANALYSIS OF LOCAL COMPOSITE QUANTILE REGRESSION IN DIFFUSION MODELS

Dr. Wei Li
College of Mathematics and Information Science, Henan Normal University, 453007, Henan Province, P. R. China.

Abstract: *In this paper, we delve into the realm of Composite Quantile Regression (CQR) for parameter estimation within the context of diffusion models. While CQR has found utility in classical linear regression models and general non-parametric regression models, it has yet to be explored extensively in the domain of diffusion models. The diffusion model we consider operates within the framework of a filtered probability space (Ω, F, (Ft)t≥0, P), described by the stochastic differential equation:*
dXt = β(t)b(Xt)dt + σ(Xt)dWt,
where β(t) is a time-dependent drift function, σ(·) and b(·) are known functions. Notably, this model encompasses several renowned option pricing models and interest rate term structure models, including Black and Scholes (1973), Vasicek (1977), Ho and Lee (1986), and Black, Derman, and Toy (1990), among others.
Our exploration of CQR in diffusion models seeks to provide a robust framework for estimating regression coefficients in scenarios with intricate dynamics. By extending CQR to this domain, we aim to enhance our understanding of parameter estimation in diffusion models and contribute valuable insights to financial modeling and related fields.

Keywords: *Composite Quantile Regression, Diffusion Models, Parameter Estimation, Financial Modeling, Stochastic Differential Equation.*

1. Introduction

Composite quantile regression (CQR) is proposed by Zou and Yuan (2008) for estimating regression coefficients in classical linear regression models. More recently, Kai el.(2010) considers a general non-parametric regression models by using CQR method. However, to our knowledge, little literature has researched parameter estimation by CQR in diffusion models. This motivates us to consider estimating regression coefficients under the framework of diffusion models. In this paper, we consider the diffusion model on a filtered probability space (□,F,(F*t*)*t*□0,*P*) ,

(1.1) *dX*$_t$ □□(*t*)*b*(*X* $_t$)*dt* □□(*X* $_t$)*dW*$_t$,

□(*t*) *W*^{t} is the standard Brownian motion. *b*(□) and□(□) are known where is a time-dependent drift function and functions. Model (1.1) includes many famous option pricing models and interest rate term structure models, such as Black and Scholes(1973), Vasicek(1977), Ho and Lee(1986), Black, Derman and Toy (1990) and so on. □(*t*)

We allow being smooth in time. The techniques that we employ here are based on local linear fitting (see Fan and Gijbels(1996)) for the time-dependent parameter. The rest of this paper is organized as follows. In Section 2, we propose the local linear composite quantile regression estimation for the drift parameter and study its asymptotic properties. The asymptotic relative efficiency of the local estimation with respect to local least squares estimation is discussed in Section 3. The proof of result is given in Section 4.

2. Local estimation of the time-dependent parameter

{X ti ,i □ 1,2,□,n □1} t1 □ t2 □□□ tn□1. Denote
Let the data be equally sampled at discrete time points,
Yti □ Xti□1 □ X ti ,□ti □Wti□1 □Wti , and □i □ ti□1 □ti. Due to the independent increment property of Brownian motion
Wt ,□ti are independent and normally distributed with mean zero and variance[1i]. Thus, the discretized version of the model (1.1) can be expressed as
(2.1) Yti □ □(ti)b(X ti)□i □□(X ti)□iZti
Zti 1/ □ i .The first-order discretized
where are independent and normally distributed with mean zero and variance approximation error to the continuous-time model is extremely small according to the findings in Stanton (1997) and Fan and Zhang(2003), this simplifies the estimation procedure.
Suppose the drift parameter □(t) to be twice continuously differentiable in t . We can take □(t) to be local t o , we use the approximation linear fitting. That is, for a given time point
(2.2) □(t) □□(t0) □□'(t0)(t □t0)
for t in a small neighborhood of to . Let h denote the size of the neighborhood and K(□) be a nonnegative weighted function. h and K(□) are the bandwidth parameter and kernel function, respectively. Denoting
□0 =□(to) and
□1 □□'(to) , (2.2) can be expressed as
(2.3) □(t) □□0 □□1(t □t0) .
□(t)
Now we propose the local linear CQR estimation of the drift parameter . Let
k
□k =____
□□k (r) □□kr □□I{r□0},k □ 1,2,□,q ,which are q check loss functions at q quantile positions: q □ 1 . Thus,
□(t)
following the local CQR technique, can be estimated via minimizing the locally weighted CQR loss
q n Y ti □1
(2.4) □ {□□□k { [b(Xti)] □□0k □□1(ti □t0)}Kh(ti □t0)}
ti □t 0) Kh (ti □to)=K(where h and h is a properly selected bandwidth. Denote the minimizer of the locally weighted
(□ˆ01,□ˆ02,□,□ˆ0q,□ˆ1)T
CQR loss (2.4) by . Then, we let
q
(2.5) □ˆ(t0) □ 1 □□ˆ 0k
q k□1

We refer to □ˆ(to) as the local linear CQR estimation of□(to) , for a given time point t o . To obtain the
□ˆ(□) estimated function , we usually evaluate the estimations at hundreds of grid points.
In order to discuss the asymptotic properties of the estimation, we introduce the following assumptions. Throughout this paper, M denotes a positive generic constant independent of all other variables.

---

[1] k□1 i□1 i ,

$b(\square)$ $\square(\square)$

(A1) The functions and in model (1.1) are continuous.

$K(\square)$

(A2) The kernel function is a symmetric and Lipschitz continuous function with finite support $[\square M, M]$

.

$h=h(n) \square 0$ $nh \square 0$.

(A3) The bandwidth and $F(\square)$ $f(\square)$

Let and be the cumulative density function and probability density function of the error, $g(\square)$ $[a,b]$ respectively. denotes the density function of time, usually a uniform distribution on time interval . Define

$\square_j \square \square u^j K(u)du$, $\square_j \square \square u^j K^2(u)du$, $j \square 1,2,\square$

and

1 q q $\square$ kk'

(2.6) $R(q) \square_2 \square\square$

q $k \square 1$ $k' \square 1$ $f(ck) f(ck')$

$ck \square F^{\square 1}(\square k)$ and $\square kk' = \square k \square \square k' \square\square\square k$ k'. where

$\square\hat{}(t_0)$

Theorem 2.1 Under assumptions (A1)-(A3), for a given time point t_0, the local CQR estimation from (2.5) satisfies,

(2.7) $E[\square\hat{}(t_0)] \square \square(t_0) \square \frac{1}{2} \square''(t_0)\square_2 h^2 \square o(h^2)$

2

(2.8) $Var[\square\hat{}(t_0)] \square \frac{1}{\quad} \frac{}{2} \square\square_0 (X_t)$ $R(q) \square o(\frac{1}{nh})$ nh $g(t_0)b(X_{t_0})$

and, as $n \square\square$,

2

(2.9) $\sqrt{nh}\{\square\hat{}(t_0) \square \square(t_0) \square \frac{1}{2} \square''(t_0)\square h^2\} \square_L N(0, \frac{}{2} \square\square_0$ (X_t) $R(q))$

2 $g(t_0)b(X_{t0})$

$\square_L$ means convergence in distribution.

where

3. Asymptotic relative efficiency

We discuss the asymptotic relative efficiency(ARE) of the local linear CQR estimation with respect to the local linear least squares estimation(see Fan and Gijbels(1996)) by comparing their mean-squared errors(MSE).From

$\square\hat{}(t_0)$. That is, theorem 2.1, we obtain the MSE

2

(3.1) $MSE[\square\hat{}(t_0)] \square [\frac{1}{2} \square''(t_0)\square_2]^2 \square \frac{1}{\quad} \frac{}{2} \square\square_0$ (X_t) $R(q) \square o(h^4 \square \frac{1}{nh})$

2 nh $g(t_0)b(X_{t_0})$ nh

We obtain the optimal bandwidth via minimizing the MSE (3.1), denoted by

$hopt\,(t_0)\,]\,\square\,[\,\square\square 0\,2(\,Xt0\,)R(q)\ \ \bar{}\ ]\ \bar{1}5n^{\square}5\ 1$

$2 \quad 2$

$g(t_0)b\,(\,X_t0\,)[\square''(t_0)\square_2\,]$

.

$\square(t^0)$, denoted by $\square^{\hat{}LS}(t^0)$, is The MSE of the local linear least squares estimation of

2

(3.2) $MSE[\square\hat{}_{LS}(t_0)\,]\,\square\,[^1\square''(t_0)\square_2]^2h^4\,\square\,{}^1\,\underline{\qquad\quad {}_0\quad}\square\square^0 \qquad (\,X^{\,t})\quad \square o(h^4\,\square\,{}^1)$

2

$2 \qquad nh\,g(t_0)b\,(\,X_{t0}\,) \qquad nh$

and the optimal bandwidth is

$\square\square 2(\,X\,) \quad 1 \quad \underline{\ 1 \quad {}_0 \qquad {}_{t_0}\qquad}\ \ {}^{-}_{5}\ \ {}^{\square}_{5}$

opt

$hLS\,(t0)\,]\,\square\,[\qquad 2 \qquad 2\ \ ]\,n$

$g(t_0)b\,(\,X_t0\,)[\square''(t_0)\square_2\,]$

.

By straightforward calculations, we have, as ${}^n\square\square$,

$MSE[\square\hat{}_{LS}(t_0)\,]\,\square\,[R\,'(q)]^{\square\frac{4}{5}}$

$\overline{MSE[\square\hat{}(t_0)\,]}$

Thus, the ARE of the local linear CQR estimation with respect to the local linear least squares estimation is

$\square\frac{4}{5}$

(3.3) $ARE(\square\hat{}(t_0),\square\hat{}_{LS}(t_0))\,\square\,[R(q)]$

(3.3) reveals that the ARE depends only on the error distribution. The ARE we obtained is equal to that in Kai el.(2010).

$ARE(\square\hat{}(t^0),\square^{\hat{}LS}(t^0))$ for some commonly seen error distributions. Table 1 in Kai Table 3.1 displays el.(2010) can be seen as ARE for more error distributions.

Table 3.1: Comparisons of $ARE(\square\hat{}(t0),\square\hat{}LS(t0))$ for the values of q

Error	$q\,\square 1$ $q\,\square\,5$	$q\,\square\,9$ $q\,\square\,19$	$q\,\square\,99$
$N(0,1)$	0.6968 0.9339	0.96590.9858	0.9980
Laplace	1.7411 1.2199	1.1548 1.0960	1.0296
$0.9N(0,1)$ $\square 0.1N(0,10^2)$	4.0505 4.9128	4.70693.5444	1.1379

From Table 3.1, we can see that the local linear CQR estimation is more efficient than the local linear least squares estimation when the error distribution is not standard normal distribution. When the error distribution is

N(0,1) and q □ 1,5,9,19,99 , the ARE(□^(t0),□^LS (t0)) is very close to 1, which demonstrates that the local linear
CQR estimation performs well when the error conforms to the standard normal distribution too.

4. Proof of result

□S11 S 12 □
S □□ □
In order to prove theorem 2.1, we first give some notations and lemmas. Let □S21 S 22□ , and
□□11 □ 12 □
□□□ □
□□21 □ 22□ , where S11 is a q□q diagonal matrix with diagonal elements f (ck),k □1,2,□,q ,
q
S □□ f (c)
S12 □ (□1 f (c1),□1 f (c2),□,□1 f (cq))T , S21 □ S12T and 22 2□k□1 k . □ 11 is a q□q matrix with (k,k ') -
□□0 k ,k ' ,k,k ' □ 1,2,□,q, □12 □ (□1□kq'□1□1k ' ,□1□qk '□1□2k ' ,□,□1□kq'□1□qk ')T
□21=□12 ' □22 □□2□kq,k '□1□kk '
element , and .
□□ □(X) □□
k √0k 0 k √1 0 u □ nh □□ □□(t)□ t0 c
Furthermore, let □□ b(Xt0) □□ , □,v □ h nh□□ □□' (t) □

1 □ ti □t0 □ di, k □i,k □ □uk □ ck □ □ □□r i
□ v √ □□b(Xti) b(Xt0) □□ with ri □ □(ti) □□(t0) □□' (t0
and nh □ h □ .Write)(ti □t0) .
□□ □(Xti) □(Xt0)□ □
Define i,k to be ti k i,k titi k . Let n 11 12
1q 1(q□1) with
□ 1 □n □ □,q w1(□ q□1) □ 1 □□q n □i□,kKh(ti □t0) ti □t 0
w1k □ √ □i,k Kh(ti □t0),k □1,2,
nh i□1 , and nh k□1 i□1 h.

Lemma 4.1 Under assumption (A1)-(A3), minimizing (2.4) is equivalent to minimizing the following term:
q □ n □i*,k Kh(ti □t0)□ q n □i*,kKh(ti □t0)(ti □t0) q
Ln(□) □ □uk □□□□v□□□□Bn,k (□)
k□1 □ i□1□ k□1 i□1k□1

□□ I □Z □c □d
□ □ □ □ □ T
b(X)□(X)□□ □ W □ (w ,w ,□,w ,w /)
□ $\frac{1}{2}$ □T Sn□□ (Wn*) T □□ op (1)
□=(u ,u ,□,u ,□)
1 1 q with respect to , where
□

1 □□X t □ √nh h√nh

Bn,k □□□□□*i*□*n*1 □□□□*Kh* □*ti* □*t*0 □□*i*,1 □□□□*I*□□ *ti* *k* *d*□ *i*,1*b*□*Xti* *ti* □ *zb*□*Xttii* □□□ - *I*□□ *ti* *k* *di*,1*b*□*Xti ti* □□□□□□□*dz*□ □□□ *Sn* □□□□*SSnn*,,1121 *SSnn*,,1222 □ □□□,

0 □ □*Z* □*c* □

□ □□ □*X* □ □□□*X* □□□□ □□□*Z* □*c* □ □□*X* □ □□□□□ □, □

□ *n* *b*□*Xti* □ □

Sn,11 □□□*Kh* □*ti* □*t*0 □ □ *i* □□*S*11 with □□ *i*□1 *nh*□ *X t* □□ , *Sn*,21 □*SnT* ,12 ,

Sn,12 □□□□*n Kh* □*ti* □*t*0 □*ti* □*t*0 *b*□*X* □*ti* □*i* □□□□*f* □*c*1 □, *f* □*c*2 □,□, *f* □*cq* □□*T*

□□ *i*□1 *h* *nh*□ *X t* □□,

□*q* □*ck* □□*n* □□*Kh* □*ti* □*t*0 □(*ti* □2*t*0)2 *b*□*X*□*ti* □*i* □□ □

Sn,22 □ *f*

and □ *h* *nh*□ *Xt* □ □.

k□1 *i*□1 □

The proof of lemma 4.1 is similar to lemma 2 and lemma 3 in Kai el.(2010).

Proof of theorem 2.1

Using the results of Parzen(1962), we have

1 *n* □*ti* □*t*0□ *j*

□*Kh* □*ti* □*t*0□ *j* □*P g*□*t*0□*u j nh i*□1 *h*

□ *P* means convergence in probability. Thus,

where

g□*t*0□*b*□*X*t0 □ *g*□*t*0□*b*□*X*t0 *S*12 □

□□*S*11 □ *S*

Sn □*P S* □ □ 22□

□□*X t*0 □ □□*Xt*0 □ □*S*21 .

According to lemma 4.1, we have

L□□□□□ 1*g*□*t*1 □ *b*□*X* t0 □□*TS*□□□*Wn** □*T*□□ o_p □1□

Ln□□□□□□*Wn**□*T*□ converges in probability to the convex function Since the convex function

1 *g*□*t*0 □*b*□*X t*0 □ *T*

□ *S*□

2 □□*X t* □

0 , according to the convexity lemma in Pollard(1991), for any compact set, the quadratic

L□□□□ □

1 .

approximation to holds uniformly for . Thus, we have

ˆ □□ g□to □b□X to □ S□1Wn* □o p □1□
□n

□□X t □
o .

Define □i,k □ I□zti □ck □ □□k and Wn □ □w11,w12,□w1q,w1□q□1□□T with
w1k □ 1 □n □i,kKh □ti □to□,k □1,2,□,q w1□q□1□ □ 1 □q □n □i,k Kh □ti □to □ ti □to
nh i□1 , and nh k□1 i□1 h .

By using the central limit theorem and the Cramer-Wald theorem, we have
Wn □E(Wn) N (0,I)
(4.1) □
L (q□1)□(q□1) Var(Wn)
.

Notice that Cov(□i,k ,□i,k ') □□kk ' andCov(□i,k ,□j,k') □ 0 Ifi □ j . We have

1 n 2 (ti □to) j
□Kh (ti □to) j □P g(to)vj. nh i□1 h

Var(W) □ g(t)□. W □ N(0,g(t)□)

Thus, n 0 . Combining the result (4.1), we have n L 0 . Moreover, we have
1 n
* 2 *
Var(w1k □w1k) □ □Kh (ti □to)Var(□i,k □□i,k)
nh i□1
1 n 2 | di,k | b(Xti)
□ □Kh (ti □to)[F(ck □) □F(ck)] □□p (1) nh i□1 □(Xti)
And
n q
* 1 2 ti □t o *
Var(w1(q□1) □ w1(q□1)) □ □Kh (ti □to) Var(□□i,k □□i,k)

nh i□1 h k□1

q2 n 2 ti □t o | di,k | b(Xti)
□ □Kh (ti □to) maxk[F(ck □) □F(ck)] □□p (1). nh i□1h □(Xti)

Var(wn* □ wn) □□ p *
Therefore, (1) . Using Slutsky's theorem yields wn □L N (0, g(to)□).

Thus,

□ □(Xt0) □1 * □2 (Xt) □1 □1
□n □ S E(Wn) □L N(0, 0/2 S □S) g(t0)b(Xt) g(t0)b (Xt)

0 0

So the asymptotic bias of □^(tO) is:

bias(□^(tO)) □ 1 □(Xt0) □q ck □ 1 □(Xt0) eqT□1(S11)□1E(W1 * n) q b(Xt0) k□1 q nh g(t0)b(Xt0)

□ 1 □(Xt0) □q ck □ 1 □(Xt0) □n Ki □q 1 □□F(ck □ di,kb(Xt0)) □ F(ck)□□ , q b(Xt0) k□1 q nh g(t0)b(X t0) i□1 k□1 f (ck) □□ □(X ti) □□ where

Ki □ Kh(ti □t0),eq□1 □ (1,1,□, 1)T and W1*n □(w11* ,w12* ,...w1*q)T .

q
c □ 0
z □ k , and
Note that ti is symmetric, thus k□1

1 q 1 □ di,kb(Xti) □ rbi (Xti)
□ □F(ck □) □ F(ck)□□□ (1□oP(1)). q k□1 f (ck) □□ □(Xt) □□ □(Xt)

i i

Therefore,

1 □(Xt0) n rbi (X ti) Since
bias(□^(t0)) □ □Ki (1□oP (1)).
nh g(t0)b(X t0) i□1 □(X ti)
n rb(X) g(t)□" (t)b(X)
1
□Ki i ti □ 0 0 t0 □2 h2 (1□oP (1)). We have
nh i□1 □(X ti) 2□(X t0)

1/2

bias(□^(t0)) □□" (t0)□2h2 □oP (h2). and
□2 (X)

$Var[\square \hat{}(t_0)] \square \frac{1}{2} t_0 12\, e_q T \square 1 (S \square 1 \square S \square 1)_{11} e_q \square 1 \square o_p(1)\ nh\, g(t_0) b(X_{t_0})\ q\ nh$

$\square 1 \frac{}{2} v_0 \square 2 (X_t)\ \ R(q) \square o_p(1). \quad nh\, g(t_0) b(X_{t_0})\ nh$

This completes the proof.

References

Black, F., Derman, E. and Toy, W.(1990).*A one-factor model of interest rate and its application to treasury bond options*. Finan. Analysts' J., **46:**33-39.

Black,F. and Scholes, M. (1973). *The pricing of options and corporate liabilities*. J. Polit. Economy, **81:**637-654.

Fan, J. and Gijbels, I.(1996).*Local polynomial modelling and its applications*. Chapman and Hall, London.

Fan, J., Jiang, J., Zhang, C. and Zhou, Z.(2003).*Time-dependent diffusion models for term structure dynamics and the stock price volatility*. Statistica Sinica, **13:**965-992.

Kai,B.,Li,R. and Zou, H.(2010).*Local composite quantile regression smoothing: an efficient and safe alternative to local polynomial regression*.J.R. Statist. Soc. B,**72:**49-69.

Parzen, E.(1962).*On estimation of a probability density function and model*. Ann. Math. Statist., **33:**1065-1076.

Pollard, D.(1991).*A symptotics for least absolute deviation regression estimations}.Econometr*.Theory,**7:**186-199.

Stanton,R.(1997).*A nonparametric model of term structure dynamics and the market price of interest rate risk*. J. Finance, **52:**19732002.

Vasicek,O.A.(1977).*An equilibrium characterization of the term structure*. J. Finan. Econom., **5:**177-188.

Zou,H. and Yuan, M.(2008).*Composite quantile regression and the oracle model selection theory*.Ann. Statist.,**36:**1108-1126. T.A.Louis, *Finding observed information using the EMl gorithm*, J.Royal Stat.Soc, 1982, B44: 226-233.