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EXPLORING NONZERO CENTRAL IDEMPOTENTS WITHIN PRE-HILBERT ALGEBRA STRUCTURES

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Abstract: *In this study, we explore the properties of pre-Hilbert and absolute valued algebras, shedding light on their fundamental characteristics. A real algebra is considered a pre-Hilbert algebra when its norm is derived from an inner product. On the other hand, absolute valued algebras are those whose norms satisfy the equality condition $ab = a\ b$ for all $a,b \in A$. We investigate the relationships between these algebraic structures, and we extend Rodriguez's theorem to more general scenarios.*

Specifically, we demonstrate that if a two-dimensional real algebra is considered, it can be isomorphic to a new class of two-dimensional pre-Hilbert algebras. Furthermore, when dealing with algebraic algebras containing nonzero central idempotents, we establish equivalences between the flexibility of the algebra, its degree, and its orthogonality properties within certain vector spaces.

This research contributes to a deeper understanding of the algebraic structures and their interplay, providing insights into the fundamental properties of pre-Hilbert and absolute valued algebras.

Keywords: *pre-Hilbert algebras, absolute valued algebras, algebraic algebra, orthogonality, flexibility.*

1 – Introduction

Let A be a non-necessarily associative real algebra which is normed as real vector space. We say that a real algebra is a pre-Hilbert algebra, if it's norm . come from an inner product $(.|.)$, and it's said to be absolute valued algebras, if it's norm satisfy the equality $ab = a\ b$, for all $a,b \in A$. We recall that the set of pre-Hilbert absolute valued algebras is contained in the set of pre-Hilbert algebras satisfying the identity $a^2 = a^2$ for all $a \in A$. Note that, the norm of any absolute valued algebras containing a nonzero central idempotent (or finite dimensional) come from an inner product [2] and [3]. We assume that A is pre-Hilbert algebra, without divisors of zero and satisfying $a^2 = a^2$ for all $a \in A$. An interesting Rodriguez's theorem [6] assert that every two-dimensional real absolute valued algebra is isomorphic to $\mathbb{C}, \mathbb{C}^*, {}^*\mathbb{C}$ or $\overset{*}{\mathbb{C}}$ (the real algebras obtained by endowing the space $\mathbb{C}$ with the product $x * y = \bar{x}y, x * y = x\bar{y}$, and $x * y = \bar{x}\,\bar{y}$ respectively). We extend the above mentioned theorem to more general situation, indeed, we prove that if A has dimension two, then A is isomorphic to a new classes of two-dimensional pre-Hilbert algebras (section 3). Also we show, in section 4, that if A is algebraic algebra and contains a nonzero

central idempotent f such that $fa = a$ and $a^2 \leq a^2$ for all $a \in A$, then the following assertion are equivalent:
i) A is flexible.
ii) A has degree two and if $\{f, u, v\}$ is an orthogonal family, then $\{f, u, v, uv\}$ is too, where $u,v \in V := \{w \in A | (w|f) = 0\}$.
And the counter example is given. Moreover, we prove that if A contains a nonzero central idempotent f such that $fa = a$ and $a^2 \leq a^2$ for all $a \in A$, then the following statements are equivalent:
1. A is power commutative 2. A is third power associative
3. A is algebraic of degree two.

2. – Notation and preliminaries results

In this paper all the algebras are considered over the real numbers field $\mathbb{R}$.

Definition 2.1 Let B be an arbitrary algebra.
i - B is called flexible, if it's satisfy the identity $(x,y,x) = 0$ for all $x,y \in B$ (where $(.,.,.)=0$ denote the associatore).
ii - We say that B is third power associative, if it's satisfy the identity $(x,x,x) = 0$ for all $x \in B$. iii - B is said power commutative if any sub-algebras generated by a single element is commutative.
iv - B is called a division algebra if the operators L_x *and* R_x of left and right multiplication by x are bijective for all $x \in B \backslash \{0\}$.
v - An element a in B is said to be algebraic (of degree n) if the sub-algebra $B(a)$ generated by a is finite dimensional (of dimension n). We say that B is algebraic if all its elements are algebraic. B is said to be algebraic of bounded degree if there exist a non-negative integer number n such that $dim\, B \leq m$ for any element a in B. If this is the case, then the small such number m is called the degree of B. Clearly every finite-dimensional algebra is algebraic of bounded degree. The (1,2,4,8) theorem show that the degree of every finite-dimensional real division algebra is 1,2,4 *or* 8.

We need the following relevant results:

Proposition 2.2 |1| If $\{x_i\}$ si a set of commuting element in a flexible algebra A over a field characteristic not two.
Then the sub-algebra generated by the $\{x_i\}$ is commutative.

Theorem 2.3 [5]. Let A be a real commutative algebraic algebra without divisors of zero, then $dim\, A \leq 2$. **Lemma 2.4** [7]. Every algebra in which $x^2 = 0$ only if $x = 0$, contains a nonzero idempotent.

3 – Two dimensional pre-Hilbert algebras, satisfying $\|a^2\| = \|a\|^2$

Firstly, we would like to consider the general situation of a two-dimensional real algebra A. let $\{e_1, e_2\}$ be a basis of A and $\alpha, \beta, \lambda, \mu, \alpha', \beta', \lambda', \mu' \in \mathbb{R}$. The product in A is determined by the multiplication table

	e_1	e_2
e_1	$\alpha e_1 + \beta e_2$	$\lambda e_1 + \mu e_2$
e_2	$\alpha' e_1 + \beta' e_2$	$\lambda' e_1 + \mu' e_2$

(1)

Theorem 3.1. The algebra A determined by the table **(1)** is division algebra if and only if 1) $4(\alpha\mu - \beta\lambda)(\alpha'\mu' - \beta'\lambda') > (\alpha'\mu + \alpha\mu' - \beta'\lambda - \beta\lambda')^2$.
2) $4(\alpha\beta' - \beta\alpha')(\lambda\mu' - \mu\lambda') > (\alpha'\mu - \alpha\mu' - \beta'\lambda + \beta\lambda')^2$.

Proof. Let $\{e_1, e_2\}$ be a basis of A such that the multiplication of A is given by the table **(1)**. Then for an arbitrary element $a = xe_1 + ye_2$ in A, we have
$L_a(e_1) = (x\alpha + y\alpha')e_1 + (x\beta + y\beta')e_2$ and $L_a(e_2) = (x\lambda + y\lambda')e_1 + (x\mu + y\mu')e_2$
So the matrix of L_a in the above basis can be expressed as follow
$$M_{L_a} = \begin{pmatrix} x\alpha + y\alpha' & x\lambda + y\lambda' \\ x\beta + y\beta' & x\mu + y\mu' \end{pmatrix}$$

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We have
$det(M_{La}) = x^2(\alpha\mu-\beta\lambda)+xy(\alpha'\mu+\alpha\mu'-\beta'\lambda-\beta\lambda')+y^2(\alpha'\mu'-\beta'\lambda')$
So A is a division algebra if and only if $det(M_{La})\neq 0$, which is equivalent to $4(\alpha\mu-\beta\lambda)(\alpha'\mu'-\beta'\lambda')>(\alpha'\mu+\alpha\mu'-\beta'\lambda-\beta\lambda')^2$.
By the same way we have the right multiplication of a, R_a is invertible if and only if $det(M_{Ra})\neq 0$, which is equivalent to $4\ \alpha\beta' - \beta\alpha'\ \ \lambda\mu' - \mu\lambda'\ > \ \alpha'\mu - \alpha\mu' - \beta'\lambda + \beta\lambda'\ ^2$. □
Let $A_1(\gamma,\delta)$,$A_2(\gamma,\delta)$, and $A_3(\gamma,\delta)$, be the real pre-Hilbert algebras defined by the multiplication tables **(2)**, **(3)** and **(4)** respectively, with $(\gamma,\delta)\in\mathbb{R}\times\mathbb{R}^*$ ($\mathbb{R}^*$: the set of nonzero real numbers). And let $\{e,u\}$ be an orthonormal basis (where e is a nonzero idempotent)

	e	u
e	e	$\gamma e+\delta u$
u	$-\gamma e-\delta u$	e

(2)
$A_1(\gamma,\delta)$

	e	u
e	e	$\gamma e+\delta u$
u	$-\gamma e+(2-\delta)u$	$-e$

3
$A_2(\gamma,\delta)$

	e	u
e	e	$\gamma e+\delta u$
u	$-\gamma e-(2+\delta)u$	$-e$

4
$A_3(\gamma,\delta)$

Remark 3.2 i) The real algebra given by table **(2)** is a division algebra for all $(\gamma,\delta)\in\mathbb{R}\times\mathbb{R}^*$ ii) The real algebra given by table **(3)** is a division algebra, if and only if, $\gamma^2+\delta^2<2\delta$ iii) The real algebra given by table **(4)** is a division algebra, if and only if $\gamma^2+\delta^2<-2\delta$

Proof. Consequence of the theorem 3.1 □

Lemma 3.3 The algebras $A_1(\gamma,\delta)$,$A_2(\gamma,\delta)$, and $A_3(\gamma,\delta)$ satisfies the identity $a^2 = a\ ^2$ for all $a\in A$ and $(\gamma,\delta)\in\mathbb{R}\times\mathbb{R}^*$.

Proof. According to remark 3.2, $A_1(\gamma,\delta)$, with $\gamma,\delta\in\mathbb{R}^*$, is a two-dimensional real division algebra. And let $a\in A_1(\gamma,\delta)$, can be written as $a = e+u$ (where $\{e,u\}$ is an orthonormal basis of $A_1(\gamma,\delta)$). So by a simple calculation we have $a^2 = a\ ^2$, similarly proof for the others cases $A_2(\gamma,\delta)$, and $A_3(\gamma,\delta)$. □

Lemma 3.4 Let A be a real pre-Hilbert algebra, without divisors of zero, and satisfying $a^2 = a\ ^2$ for all $a\in A$. Then the following equalities hold for all orthogonal elements $x,y\in A$:
1) $(x^2|xy+yx) = 0$
$\|xy+yx\|^2+2(x^2|y^2) = 2\|x\|^2\|y\|^2$
2)

Proof. The equality $\|x^2\|^2 = (\|x\|^2)^2$ gives meaning to a polynomial p with real coefficients of degree ≤ 3 in λ,
identically null, such that:
$P(\lambda) = 2(x^2|xy+yx)\lambda^3+(\|xy+yx\|^2+2(x^2|y^2)-2\|x\|^2\|y\|^2)\lambda^2+2(y^2|xy+yx)\lambda.$

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Thus 1) $(x^2|xy + yx) = 0$
$\|xy + yx\|^2 + 2(x^2|y^2) = 2\|x\|^2\|y\|^2$
2). □

Now we can state our main result in this section

Theorem 3.5 Let A be a two-dimensional real pre-Hilbert algebra, without divisors of zero, and satisfying $\|a^2\| = \|a\|^2$ for all $a \in A$. Then, A is isomorphic to $A_1(\gamma,\delta)$,$A_2(\gamma,\delta)$, or $A_3(\gamma,\delta)$ for all $(\gamma,\delta) \in \mathbb{R} \times \mathbb{R}^*$, such that $\gamma^2 + \delta^2 < \pm 2\delta$.

Proof. According to lemma 2.4, A is a two-dimensional real division algebra, containing a nonzero idempotent e. And let $\{e,u\}$ be an orthonormal basis of A. Then there exists $\gamma,\gamma' \in \mathbb{R}$ and $\delta,\delta' \in \mathbb{R}^*$, such that $eu = \gamma e + \delta u$ and $ue = \gamma' e + \delta' u$.
We have $eu+ue=(\gamma+\gamma')e+(\delta+\delta')u$, which means by lemma 3.4 (1) That $\gamma = -\gamma'$ and $eu + ue = (\delta + \delta')u$. Since
$0 = (u^2|eu + ue) = (\delta + \delta')(u^2|u)$,
Then $eu + ue = 0$ or $u^2 = \pm e$, we distinguish the following cases.
case 1: If $ue + eu = 0$, then by lemma 3.4 (2) we have $(e|u^2) = 1$, so $\|u^2 - e\|^2 = 2 - 2 = 0$. Consequently $u^2 = e$, thus A is isomorphic to $A_1(\gamma,\delta)$.
case 2: If $u^2 = -e$, then $(\delta + \delta')^2 = 4$ (lemma 3.4 (2)). That is $\delta + \delta' = 2$ *or* $\delta + \delta' = -2$ i) If $\delta + \delta' = 2$ then $\delta' = 2 - \delta$. So A is isomorphic to $A_2(\gamma,\delta)$.
ii) If $\delta+\delta' = -2$ then $\delta' = -2-\delta$. So A is isomorphic to $A_3(\gamma,\delta)$. □
We get the following results.
Corollary 3.6 Let A be a two-dimensional real pre-Hilbert algebra, containing a nonzero central idempotent e, without divisors of zero and satisfying $\|a^2\| = \|a\|^2$ for all $a \in A$. Then A is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$
Proof. According to theorem 3.5, the algebra A is isomorphic to $A_1(\gamma,\delta)$,$A_2(\gamma,\delta)$, or $A_3(\gamma,\delta)$ for all $(\gamma,\delta) \in \mathbb{R}\times\mathbb{R}^*$, such that $\gamma^2 + \delta^2 < \pm 2\delta$. Since e is a central idempotent, then we have the following cases:
i) If A is isomorphic to $A_1(\gamma,\delta)$, then $eu = -ue$ which is absurd.
ii) If A is isomorphic to $A_2(\gamma,\delta)$, then $\gamma = 0$ and $\delta = 1$. So A is isomorphic to $\mathbb{C}$
$*$ iii) If A is isomorphic to $A_3(\gamma,\delta)$, then $\gamma = 0$ and $\delta = -1$. So A is isomorphic to $\mathbb{C}$. □
Corollary 3.7 Let A be a two-dimensional real third power associative pre-Hilbert algebra, without divisors of zero and satisfying $\|a^2\| = \|a\|^2$ for all $a \in A$. Then A is isomorphic to $\mathbb{C}$ or $\mathbb{C}$.
Proof. According to theorem 3.5, the algebra A is isomorphic to $A_1(\gamma,\delta)$,$A_2(\gamma,\delta)$, or $A_3(\gamma,\delta)$ for all $(\gamma,\delta) \in \mathbb{R}\times\mathbb{R}^*$, such that $\gamma^2 + \delta^2 < \pm 2\delta$. The identity $(u,u,u) = 0$ imply that $eu = ue$, so e is a nonzero central idempotent. We conclude that A is isomorphic to $\overset{*}{\mathbb{C}}$ or $\mathbb{C}$ (Corollary 3.6). □
Now we conclude the theorem of A. Rodriguez
Corollary 3.8 Let A be a two-dimensional real absolute valued algebra. Then A is isomorphic to $\mathbb{C}, \mathbb{C}^*, *\mathbb{C}$ or $\overset{*}{\mathbb{C}}$.
Proof. Since A is a finite-dimensional real absolute valued algebra, then A satisfying $\|a^2\| = \|a\|^2$ for all $a \in A$. and it's norm comes from an inner product [2]. Using theorem 3.5, the algebra A is isomorphic to $A_1(\gamma,\delta)$,$A_2(\gamma,\delta)$, or
$A_3(\gamma,\delta)$for all $(\gamma,\delta) \in \mathbb{R}\times\mathbb{R}^*$, such that $\gamma^2 + \delta^2 < \pm 2$. We have
$(ue|e) = \pm(ue|u^2) = \pm(u|e) = 0$ and $(eu|e) = \pm(eu|u^2) = \pm(u|e) = 0$
This imply that the two elements ue *and* u (respectively eu *and* u) are linearly dependent, thus $\gamma = 0$. Therefore i) If A is isomorphic to $A_1(\gamma,\delta)$, then the identity $eu = -ue = \pm u$ imply that A is isomorphic to $\mathbb{C}^*$ or $*\mathbb{C}$ ii) If A is isomorphic to $A_2(\gamma,\delta)$, then the identity $\|eu\| = \|ue\| = \|u\| \|e\| = 1$, imply that, $\delta = 1$ which

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means that A is isomorphic to $\mathbb{C}$. iii) If A is isomorphic to $A_3(\gamma,\delta)$, then the identity $eu = ue = u$ $e=1$, imply that, $\delta=-1$ which means that A is isomorphic to $\mathbb{C}$. ∎

4 - Pre-Hilbert algebras containing a nonzero central idempotent f such that $fa = a$ and $\|a^2\| \le \|a\|^2$

We begin with the following preliminary results.

Proposition 4.1 Let A be real pre-Hilbert algebra containing a nonzero central idempotent f such that $fa = a$ and $\|a^2\| \le \|a\|^2$ for all $a \in A$. Then the following equalities hold:

$i)$ $\|a^2\| = \|a\|^2$

$ii)$ $a^2 - 2(a|f)fa + \|a\|^2 = 0$

Proof. i) Let $a\in A$, having an orthogonal sum decomposition $\lambda f+u$, the equality $\|a^2\|^2 \le (\|a\|^2)^2$ can be written

$$\|\lambda^2 + 2\lambda fu + u^2\|^2 \le (\lambda^2 + \|u\|^2)^2 \quad (1)$$

As $\|fx\| = \|x\|$, then $(fx|f) = (f|x)$ for all $x \in A$. The development of (1) gives

$$2\lambda^2\big((f|u^2) + \|u\|^2\big) + 4\lambda(fu|u^2) + \|u^2\|^2 - \|u\|^4 \le 0 \qquad ((f|u) = 0) \qquad (2)$$

We replace λ by $-\lambda$ we get

$$2\lambda^2\big((f|u^2) + \|u\|^2\big) - 4\lambda(fu|u^2) + \|u^2\|^2 - \|u\|^4 \le 0 \qquad (3)$$

We add (2) and (3), we have

$$2\lambda^2\big((f|u^2) + \|u\|^2\big) + \|u^2\|^2 - \|u\|^4 \le 0 \qquad (4)$$

Since (4) hold for all $\lambda\in\mathbb{R}$ and $\|u^2\|^2 - \|u\|^4 \le 0$, then $(f|u^2) \le -\|u\|^2$.
According to the Cauchy-Schwarz inequality, we have
$|(f|u^2)| \le \|f\|\|u^2\| \le \|u\|^2$.
Then $(f|u^2) = -\|u\|^2$ also $\|u^2\| = \|u\|^2$, thus

$$\begin{aligned}\|u^2 + \|u\|^2 f\|^2 &= \|u^2\|^2 + 2\|u\|^2(u^2|f) + \|u\|^4\|f\|^2 \\ &= \|u\|^4 - 2\|u\|^4 + \|u\|^4 \\ &= 0\end{aligned}$$

hence $u^2 = -\|u\|^2 f$. On the other hand $a^2 = \lambda^2 f + 2\lambda fu + u^2$, then

$$\begin{aligned}\|a^2\|^2 &= \|(\lambda^2 - \|u\|^2)f + 2\lambda fu\|^2 \\ &= (\lambda^2 - \|u\|^2)^2 + 4\lambda^2\|u\|^2 \qquad ((f|fu) = (f|u) = 0) \\ &= (\lambda^2 + \|u\|^2)^2 \\ &= \|a\|^4\end{aligned}$$

Therefore $\|a^2\| = \|a\|^2$

ii) We have

$$\begin{aligned}a^2 &= \lambda^2 f + 2\lambda fu + u^2 \\ &= -\lambda^2 f + 2\lambda f(\lambda f + u) - \|u\|^2 f \\ &= 2\lambda f(\lambda f + u) - (\lambda^2 + \|u\|^2)f \\ &= 2(f|a)fa - \|a\|^2 f.\end{aligned}$$ ∎

Remark 4.2 Let $V = \{u \in A \mid (u|f) = 0\}$, then the following equalities hold:

i) $V = \{u \in A \mid u^2 = -\|u\|^2 f\}$ ii) $uv + vu = -2(u|v)f$ for all $u,v\in V$.

iii) The product $x \wedge y = xy - (xy|f)f$ for all $x,y \in V$ endows V of an anti-commutative algebra structure

Proof.

i) According to proposition 4.1 (i). ii) Let $x,y \in V$, we have $x+y \in V$. Then $(x+y)^2 = -\|x+y\|^2 f$ thus $xy + yx = -2(x|y)f$. iii) Let $x,y \in V$, we have

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$xx\wedge y + y\wedge x = xy - (xy|f)f + yx - (yx|f)f$
$= xy + yx - (xy + yx|f)f$
$= -2(x|y)f + 2(x|y)f$
$= 0$ □

Lemma 4.3. Let A be a real pre-Hilbert algebraic algebra of degree two, without divisors of zero and containing a nonzero central idempotent f such that $fa = a$ and $a^2 \le a^2$ for all $a \in A$. Then $A(f,v) = A(v)$ is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$, for every $v \in V$.

Proof. It suffices to prove that $fv = v$ *or* $fv = -v$ for every $v \in V$, without loss of generality we will assume that
$v \neq 0$, according to remark 4.2 (i), $v^2 = -\|v\|^2 f$. since A is algebraic algebra of degree two and satisfying $a^2 = a^2$ for all $a \in A$ (proposition 4.1 (i)), then by corollary 3.6 A is power-commutative algebra. It is well known that $(aa^2)a = a(a^2a)$ for any element a in A, hence
$v(v^2v) = -\|v\|^2 v(fv)$ *and* $(v^2v)v = -\|v\|^2 (fv)v$.
So $v(fv) = (vf)v$, *as* $\|vf\| = \|fv\| = \|v\|$ *and* $(fv|f) = (vf|f) \overset{*}{=} (v|f) = 0$, then $(vf)^2 = v^2 = -\|v\|^2 f$. *So* $fv = v$ *or* $fv = -v$, thus $A(v)$ is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$. □

In the next theorem we give some conditions implying that the real pre-Hilbert algebra to be flexible algebra.

Theorem 4.4. Let A be a real pre-Hilbert algebraic algebra, without divisors of zero and containing a nonzero central idempotent f such that $\|fa\| = \|a\|$ *and* $\|a^2\| \le \|a\|^2$ for all $a \in A$. Then the following assertions are equivalent:

i) A is flexible;

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ii) A has degree two and if $\{f,u,v\}$ is an orthogonal family, then $\{f,u,v,uv\}$ is too, where $u,v \in V$.

Proof. $i)\Rightarrow ii)$ Assume that A is flexible algebra, according to proposition 2.2 and theorem 2.3, A has degree two. Let $w := uv - (uv|f)f - (uv|u)u - (uv|v)v$, where $u,v \in V$, without loss of generality we assume that
$w \neq 0$. We have
$(w|f) = (w|u) = (w|v) = 0$
Since A is flexible and $uv + vu = 0$ (remark 4.2 (ii)), then
$0 = uw + wu = 2(uv|u)f + 2(uv|f)u$
This gives us $(uv|u) = (uv|f) = 0$, similarly $(uv|v) = 0$. Thus $\{f,u,v,uv\}$ is an orthogonal family.
$ii)\Rightarrow i)$ Let $x,y \in V$ such that $\|x\| = \|y\| = 1$ and $z := y - (y|x)x$, without loss of generality we can assume that $z \neq 0$, wehave $(z|f) = (z|x) = 0$. Then
$0 = (xz|x)$
$= (x(y - (y|x)x)|x)$
$= (xy + (y|x)f|x)$ $\quad (x^2 = -f, (remark\ 4.2\ (i)) \quad = (xy|x)$
The equality $fa = a$ imply that $(fa|f) = (f|a)$, for all $a \in A$. On the other hand
$0 = (xz|f)$
$= (x(y - (y|x)x)|f)$
$= (xy + (y|x)f|f)$
$= (xy|f) + (x|y)$
This means that $(xy|f) = -(x|y)$, in the same way, we get $(yx|f) = -(x|y)$. Moreover, we have
$(xy)x - x(yx) = (x\wedge y + (xy|f)f)x - x(y\wedge x + (yx|f)f)$
$= (x\wedge y)x + x(x\wedge y) + ((xy|f)f - (yx|f)f)x$
$= -2(x|x\wedge y)f + ((xy|f)f - (yx|f)f)x \quad (remark\ 4.2\ (ii)) \quad = ((xy|f)f - (yx|f)f)x$.

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$= 0$

Since A is algebraic algebra of degree two, then the sub-algebras $A(f,x)$ and $A(f,y)$ are of dimension two and isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$(lemma 4.3). We have the following cases:

1) If f is the only idempotent of A, then $xf = fx = x$ for all $x \in A$. Which means that A is a unit algebra, hence for all $a = \lambda f + x$ and $b = \gamma f + y$ in A, we have

$$(ab)a - a(ba) = [(\lambda f + x)(\gamma f + y)](\lambda f + x) - (\lambda f + x)[(\gamma f + y)(\lambda f + x)]$$
$$= (xy)x - x(yx) \qquad = 0$$ Then A is flexible.

2) If f is not unique, then $xf = fx = -x$ for all $x \in V$. Otherwise, if there exist a nonzero element $y \in V$ such that $yf = fy = y$. Then

$$f(x + y) = \pm(x + y) \qquad \text{(Lemma 4.3)}$$
$$-x + y = \pm(x + y)$$

This imply that $x = 0$ or $y = 0$, which is absurd. Therefore for all $a=\lambda f+ x$ and $b = \gamma f + y$ in A, we have

$$(a,b,a) = (\lambda f + x,\gamma f + y,\lambda f + x)$$
$$= (\lambda f,\gamma f,x) + (\lambda f,y,x) + (x,\gamma f,\lambda f) + (x,y,\lambda f) + (x,y,x)$$

Or $(x,y,x) = 0$ and $(\lambda f,\gamma f,x) + (x,\gamma f,\lambda f) = 0$. Then

$$(a,b,a) = (\lambda f,y,x) + (x,y,\lambda f)$$
$$= \lambda[(fy)x - f(yx) + (xy)f - x(yf)]$$
$$= \lambda[-yx - f(y \wedge x + (xy|f)f) + (x \wedge y + (xy|f)f)f + xy]$$
$$= \lambda[-yx + y \wedge x - x \wedge y + xy] \qquad ((xy|f) = (yx|f) = -(x| = \lambda[(x|y)f - (x|y)f]$$
$$= 0$$

Then A is flexible. □

Remark 4.5. In [4], we constructed an example of four-dimensional absolute valued algebra containing a nonzero central idempotent of degree four which is not flexible. The last imply that the condition algebraic algebra of degree two is necessary for A to be flexible.

In the rest of this section, we prove that, if A is a real pre-Hilbert algebra, without divisors of zero and contains a nonzero central idempotent f such that $fa = a$ and $a^2 \leq a^2$ for all $a \in A$. Then the following statements are equivalent:

1. A is power commutative.
2. A is third power associative.
3. A is algebraic of degree two.

We need the following preliminary result.

Lemma 4.6. Let A be a real third power associative pre-Hilbert algebra, without divisors of zero and contains a nonzero central idempotent f such that $fa = a$ and $a^2 \leq a^2$ for all $a \in A$, then $A(v)$ is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$ for every $v \in V$.

Proof. It suffices to prove that $ev = v$ or $ev = -v$ for every $v \in V$, without loss of generality we will assume that $v \neq 0$. According to remark 4.2 (i), $v^2 = -\|v\|^2 f$ as A is a third-power associative algebra. Then a linearization of the identity $(x,x,x) = 0$ gives

$$[x^2,y] + [xy + yx,x] = 0 \qquad (5)$$ where $[x,y]$ denotes the quantity $xy - yx$.

By putting $y = x^2$ in the equality (5), we get the well-known identity $(x,x^2,x) = 0$. So $(aa^2)a = a(a^2a)$ for any element a in A, hence

$v(v^2v) = -\|v\|^2 v(fv)$ and $(v^2v)v = -\|v\|^2(fv)v$

So $v(fv)= (vf)v = (fv)v$, on the other hand $\|vf\| = \|fv\| = \|v\|$ and $(fv|f) = (vf|f) = (v|f) = 0$. Then $(vf)^2 = v^2 = -\|v\|^2 f$. This imply $fv = v$ or $fv = -v$, thus $A(v)$ is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$.

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Note that in the general case, if the algebra A is power commutative. We have for every $a \in A$ the sub-algebra $A(a)$ is commutative, so $a^2a = aa^2$ which means that $(a,a,a) = 0$, then A is third power associative. □

In the next theorem we have the reciprocally case.

Theorem 4.7. Let A is a real pre-Hilbert algebra, without divisors of zero and containing a nonzero central idempotent f such that $\|fa\| = \|a\|$ and $\|a^2\| \leq \|a\|^2$ for all $a \in A$. Then the following statements are equivalent:

1. A is power commutative;
2. A is third power associative;
3. A is algebraic of degree two.

Proof. (1) ⇒(2) By definition.

(2)⇒(3) Assume that A is third power associative and let $v \in V$, we have $v^2 = -\|v\|^2 f$ (proposition 4.1 (i)). By lemma 4.6, $A(a)$ is isomorphic to $\mathbb{C}$ or $\mathbb{C}$, and consequently A is algebraic of degree two.

(3)⇒(1) Using lemma 4.3. □

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